J. Fluid Mech. (1976), *vol.* 78, *part* 3, *pp.* 577–600
Printed in Great Britain

Stokes flow past a slender body of revolution

By JAMES GEER
School of Advanced Technology, State University of New York, Binghamton

(Received 10 December 1975 and in revised form 24 May 1976)

The complete uniform asymptotic expansion of the velocity and pressure fields for Stokes flow past a slender body of revolution is obtained with respect to the slenderness ratio ϵ of the body. A completely general incident Stokes flow is assumed and hence these results extend the special cases treated by Tillett (1970) and Cox (1970). The part of the flow due to the presence of the body is represented as a superposition of the flows produced by three types of singularity distributed with unknown densities along a portion of the axis of the body and lying entirely inside the body. The no-slip boundary condition on the body then leads to a system of three coupled, linear, integral equations for the densities of the singularities. The complete expansion for these densities is then found as a series in powers of ϵ and $\epsilon \log \epsilon$. It is found that the extent of these distributions of singularities inside the body is the same for all the singular flows and depends only upon the geometry of the body. The total force, drag and torque experienced by the body are computed.

1. Introduction

We wish to discuss the slow steady motion of a viscous incompressible fluid with negligible inertial forces, i.e. Stokes flow, past a slender body of revolution. We shall assume that the body is immersed in a prescribed flow field which must satisfy the appropriate equations for Stokes flow but which is otherwise *completely* arbitrary. The slenderness ratio ϵ of the body, which is defined as the ratio of half the maximum diameter of the body to its length, will be assumed to be small. We shall obtain the complete uniform asymptotic expansion for the resulting flow field with respect to the parameter ϵ as it approaches zero. Here we are concerned only with the Stokes solution to our problem, and not the far-field (Oseen) solution.

The problem of determining Stokes flow past a slender body of revolution has been studied recently by several researchers. For example, Cox (1970, 1971) and Keller & Rubinow (1976) have presented general theories for the creeping motion of long slender bodies in a viscous fluid. They both use the method of determining inner and outer expansions and then matching the results. Batchelor (1970) has presented results for a slender body of arbitrary (not necessarily circular) cross-section in Stokes flow. Again his method is essentially that of analysing 'inner' and 'outer' flow fields. We shall not use inner and outer expansions here; instead we obtain a uniform expansion for the solution.

In §2, the basic equations and boundary conditions for Stokes flow are stated and the general problem to be solved is formulated. In §3, we first show how the general problem with arbitrary flow incident upon the body can be analysed by considering just the individual Fourier components of the incident and perturbed flow fields. Then we represent that part of the total flow due to the presence of the body as a superposition of the flows due to point singularities. The no-slip boundary condition then allows us to write down a system of three coupled, linear, integral equations, from which we shall be able to determine the unknown densities of the singularities, as well as the extent of their distributions. This idea was used by Tillett (1970), who obtained the leading terms in a uniform expansion for the special case of a uniform incident flow. We shall see that all of the integrals which appear in these equations are of the type already analysed by Geer (1975) and Handelsman & Keller (1967*a*, *b*).

In §§4 and 5, we determine the distribution of the singularities inside the body. In §4, we determine their extent by applying the criterion given by Geer (1975). In particular, we show that the extent is the same for each type of singularity we consider and is the same as that obtained in the corresponding potential-flow problem (Geer 1975). In §5, we use the expansions (given in appendix B) of the integral operators appearing in the equations to determine the asymptotic expansions for the densities of the distributions of singularities. These expansions involve integral powers of both ϵ and $\log \epsilon$.

In §6, we use the results of the previous sections to determine the complete velocity and pressure fields for the examples of a uniform flow and a purely shearing flow incident upon the body. In the final section, we derive formulae for the total force and torque experienced by the body when it is immersed in an arbitrary incident flow field.

2. Formulation of the problem

We introduce cylindrical co-ordinates (r, θ, z) in the usual way, with the z axis coinciding with the axis of the body. Let the equation of the surface of the body be described by $r \equiv (x^2+y^2)^{\frac{1}{2}} = \epsilon[S(z)]^{\frac{1}{2}}$, for $0 \leqslant z \leqslant 1$. Here $S(z)$ is a prescribed function which satisfies $\max S(z) = 1$ for $0 \leqslant z \leqslant 1$. We shall assume that $S(z)$ is regular on the interval $0 \leqslant z \leqslant 1$ with $S(0) = 0 = S(1)$ and that it can be expanded in a Taylor series about the end points as follows:

$$S(z) = \sum_{n=1}^{\infty} c_n z^n, \quad c_n = \frac{S^{(n)}(0)}{n!}, \tag{2.1}$$

$$S(z) = \sum_{n=1}^{\infty} d_n (1-z)^n, \quad d_n = \frac{(-1)^n S^{(n)}(1)}{n!}. \tag{2.2}$$

We shall assume that $c_1 \neq 0 \neq d_1$, i.e. that the radius of curvature at each end of the body is non-zero.

Now let p and $\mathbf{v} = (v_r, v_\theta, v_z)$ be the non-dimensionalized pressure and velocity of our fluid. Here v_r, v_θ and v_z are the components of the velocity field in the r, θ

and z directions, respectively. Then the equation of motion (in the absence of body forces) for Stokes flow and the continuity equation can be written as

$$\nabla p = \nabla^2 \mathbf{v}, \quad \nabla . \mathbf{v} = 0. \tag{2.3}$$

The no-slip boundary condition on the body and the condition at infinity can be written as

$$\mathbf{v} = 0 \quad \text{on} \quad r = \epsilon[S(z)]^{\frac{1}{2}} \quad \text{for} \quad 0 \leqslant z \leqslant 1, \tag{2.4}$$

and

$$\mathbf{v} \to \mathbf{v}^0, \quad p \to p^0 \quad \text{as} \quad r^2 + z^2 \to \infty. \tag{2.5}$$

In (2.4) and (2.5), $\mathbf{v}^0$ and p^0 are prescribed functions. Thus our task is to find an asymptotic expansion of the solution to (2.3)–(2.5) as $\epsilon \to 0$.

3. Derivation of the integral equations

Now let a flow field which satisfies (2.3) and which is described by a pressure p^0 and a velocity $\mathbf{v}^0$ be incident upon the body. We shall assume that p^0 and $\mathbf{v}^0$ are regular in a neighbourhood of the body. We now seek functions p^b and $\mathbf{v}^b$, which we may think of as the disturbance or perturbation pressure and velocity due to the presence of the body, that are solutions to (2.3) outside the body and that satisfy the conditions

$$\mathbf{v}^b = -\mathbf{v}^0 \quad \text{on} \quad r = \epsilon[S(z)]^{\frac{1}{2}} \quad \text{for} \quad 0 \leqslant z \leqslant 1, \tag{3.1}$$

and

$$p^b, \mathbf{v}^b \to 0 \quad \text{as} \quad r^2 + z^2 \to \infty. \tag{3.2}$$

Then $p = p^0 + p^b$ and $\mathbf{v} = \mathbf{v}^0 + \mathbf{v}^b$ will satisfy the problem formulated in §2.

In order to find a suitable representation for p^b and $\mathbf{v}^b$, we first look carefully at the general form of p^0 and $\mathbf{v}^0$. From (2.3) it follows that $\nabla^2 p^0 = 0$ in a neighbourhood of the body. Thus it follows that, for z in a neighbourhood of $0 \leqslant z \leqslant 1$ and for small r, p^0 has an expansion

$$p^0(r, \theta, z) = \tilde{A}_0(r^2, z) + \sum_{n=1}^{\infty} \{r^n \tilde{A}_n(r^2, z) \cos n\theta + r^n \tilde{B}_n(r^2, z) \sin n\theta\}. \tag{3.3}$$

In (3.3), each $\tilde{A}_n$ and $\tilde{B}_n$ is a prescribed function, regular in r^2 and z for (at least) z in a neighbourhood of $0 \leqslant z \leqslant 1$ and small values of r. By linearity and superposition, we need to consider only the case when p^0 has the form of one of the terms in (3.3), i.e. when

$$p^0(r, \theta, z) = r^n A_n(r^2, z)\, e^{in\theta} \quad (n \geqslant 0). \tag{3.4}$$

When p^0 is given by (3.4) (where $A_n(r^2, z)$ is a prescribed regular function of r^2 and z), the corresponding expressions for $\mathbf{v}^0$ can easily be found to be

$$(v_r^0, v_\theta^0, v_z^0) = -r^{n-1} e^{in\theta}(B_n(r^2, z), iC_n(r^2, z), rD_n(r^2, z)). \tag{3.5}$$

Here B_n, C_n and D_n are prescribed functions of r^2 and z. B_0 and C_0 are $O(r^2)$ as $r \to 0$. In what follows, only the case $n \geqslant 1$ will be treated in detail. The results for $n = 0$, which are similar to the results of Tillett (1970), will be given in appendix A.

Now, in order to find the functions p^b and $\mathbf{v}^b$ corresponding to the expressions in (3.4) and (3.5), we seek to represent p^b and $\mathbf{v}^b$ as the superposition of flows due to 'appropriate' point singularities distributed along a portion of the axis of the body and lying inside the body. To find these 'appropriate' (in general, higher order) singularities, we look for solutions to (2.3) which are proportional to $e^{in\theta}$ and also proportional to $(z^2+r^2)^{-k}$, where $k > 0$. By a straightforward, though tedious, calculation we find three 'singular' solutions to (2.3); they have the general form $\mathbf{v}^b = R^{-2n}r^{n-1}e^{in\theta}(\tilde{B}_n, i\tilde{C}_n, \tilde{D}_n)$ and $p^b = R^{-2n-1}r^n e^{in\theta}\tilde{A}_n$, where $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$ and $\tilde{D}_n$ take the three sets of values

$$\tilde{A}_n = 2, \quad \tilde{B}_n = R\left(\frac{r^2}{R^2} - \frac{n-2}{2n-1}\right), \quad \tilde{C}_n = \frac{2-n}{2n-1}R, \quad \tilde{D}_n = R^{-1}rz, \tag{3.6}$$

$$\tilde{A}_n = 0, \quad \tilde{B}_n = R^{-3}r^2, \quad \tilde{C}_n = -R^{-3}r^2, \quad \tilde{D}_n = R^{-3}rz, \tag{3.7}$$

$$\tilde{A}_n = 0, \quad \tilde{B}_n = R^{-1}z, \quad \tilde{C}_n = R^{-1}z, \quad \tilde{D}_n = -R^{-1}r. \tag{3.8}$$

In (3.6)–(3.8), $R \equiv (r^2+z^2)^{\frac{1}{2}}$. Of course, these solutions are only valid for $R \neq 0$.

We now represent $\mathbf{v}^b$ as the superposition of the flows described by (3.6)–(3.8), distributed with unknown densities along a portion of the z axis lying inside the body. Thus, for example, we set (for $n \geqslant 1$)

$$v_r^b(r,\theta,z,\epsilon) = e^{in\theta}\int_\alpha^\beta \left[\left(\frac{r^{n+1}}{R^{2n+1}} + \frac{(2-n)}{(2n-1)}\frac{r^{n-1}}{R^{2n-1}}\right)\tilde{p}(\xi,\epsilon) + \frac{r^{n+1}}{R^{2n+3}}\tilde{b}(\xi,\epsilon) + \frac{r^{n-1}(z-\xi)}{R^{2n+1}}\tilde{d}(\xi,\epsilon)\right]d\xi \tag{3.9}$$

with analogous formulae holding for v_θ and v_z. In (3.9), R is now given by $R = \{(z-\xi)^2+r^2\}^{\frac{1}{2}}$, while $\tilde{p}$, $\tilde{b}$ and $\tilde{d}$ are the unknown densities of the singular flows. α and β, which actually depend upon ϵ and determine the extent of the distributions, are unknown and must be found in addition to the unknown densities. They must satisfy the inequalities $0 < \alpha < \beta < 1$. The pressure p^b corresponding to (3.9) is given by

$$p^b(r,\theta,z,\epsilon) = e^{in\theta}\int_\alpha^\beta \frac{2r^n}{R^{2n+1}}\tilde{p}(\xi,\epsilon)\,d\xi. \tag{3.10}$$

The expressions for $\mathbf{v}^b$ and p^b given by (3.9) and (3.10) satisfy (2.3) outside the body and vanish at infinity. In order to determine the unknown quantities in (3.9) we use the boundary condition (3.1). When (3.9) and the analogous formulae for v_θ^b and v_z^b are used with (3.5), (3.1) becomes

$$B_n(\epsilon^2S(z),z) = \int_\alpha^\beta \left[\left\{\frac{\epsilon^2S(z)}{R^{n+\frac{1}{2}}} + \frac{2-n}{(2n-1)R^{n-\frac{1}{2}}}\right\}\tilde{p}(\xi,\epsilon) + \frac{\epsilon^2S(z)}{R^{n+\frac{3}{2}}}\tilde{b}(\xi,c) + \frac{z-\xi}{R^{n+\frac{1}{2}}}\tilde{d}(\xi,\epsilon)\right]d\xi, \tag{3.11}$$

$$C_n(\epsilon^2S(z),z) = \int_\alpha^\beta \left[\frac{2-n}{(2n-1)R^{n-\frac{1}{2}}}\tilde{p}(\xi,\epsilon) - \frac{\epsilon^2S(z)}{R^{n+\frac{3}{2}}}\tilde{b}(\xi,\epsilon) + \frac{z-\xi}{R^{n+\frac{1}{2}}}\tilde{d}(\xi,\epsilon)\right]d\xi, \tag{3.12}$$

$$D_n(\epsilon^2S(z),z) = \int_\alpha^\beta \left[\frac{z-\xi}{R^{n+\frac{1}{2}}}\tilde{p}(\xi,\epsilon) + \frac{z-\xi}{R^{n+\frac{3}{2}}}\tilde{b}(\xi,\epsilon) - \frac{1}{R^{n+\frac{1}{2}}}\tilde{d}(\xi,\epsilon)\right]d\xi, \tag{3.13}$$

with analogous equations holding for $n = 0$. Here $R \equiv \epsilon^2 S(z) + (z-\xi)^2$. Equations (3.11)–(3.13) are a set of coupled linear integral equations, from which we shall determine $\tilde{p}$, $\tilde{b}$ and $\tilde{d}$, as well as α and β.

4. Determination of α and β

We now wish to determine the uniform asymptotic expansion, as ϵ approaches zero, of the solution to (3.11)–(3.13). We shall show in the next section how this can be done by using the method of Handelsman & Keller (1967*a*) and Geer (1975).

The first step in solving (3.11)–(3.13) asymptotically is to expand both sides of each equation asymptotically as ϵ approaches zero, without taking into account the dependence of $\tilde{p}$, $\tilde{b}$ and $\tilde{d}$ upon ϵ. The left side of each of these equations can easily be expanded in a Taylor series in ϵ^2. Geer (1975) has shown how each of the integrals appearing in the right sides of (3.11) – (3.13), i.e. integrals of the form

$$I_k^j(z,\epsilon) = \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{(z-\xi)^j}{((z-\xi)^2+\epsilon^2 S(z))^{k+\frac{1}{2}}} \tilde{f}(\xi,\epsilon)\, d\xi, \tag{4.1}$$

where $j = 0, 1$ and $k = 0, 1, 2, \ldots$, can be expanded uniformly in a series involving powers of ϵ and powers of ϵ multiplied by $\log \epsilon$. In particular, it was shown that the requirement for a uniform expansion of these integrals leads to a choice of $\alpha(\epsilon)$ and $\beta(\epsilon)$ which is independent of k. Hence $\alpha(\epsilon)$ and $\beta(\epsilon)$ have asymptotic expansions as ϵ approaches zero given by (see Handelsman & Keller 1967*a*)

$$\alpha(\epsilon) \sim c_1(\tfrac{1}{2}\epsilon)^2 - c_1 c_2 (\tfrac{1}{2}\epsilon)^4 + c_1(c_1 c_3 + 2c_2^2)(\tfrac{1}{2}\epsilon)^6 + O(\epsilon^8) \tag{4.2}$$

and

$$\beta(\epsilon) \sim 1 - d_1(\tfrac{1}{2}\epsilon)^2 + d_1 d_2 (\tfrac{1}{2}\epsilon)^4 - d_1(d_1 d_3 + 2d_2^2)(\tfrac{1}{2}\epsilon)^6 + O(\epsilon^8). \tag{4.3}$$

The constants c_j and d_j which appear in (4.2) and (4.3) are defined in (2.1) and (2.2). Hence, the extent of all of our distributions of singularities is now determined.

Before we actually determine the asymptotic expansions of our singularity densities in the next section, it is convenient to rewrite our system so that the kernels are less singular as $\epsilon \to 0$. That is, by forming certain linear combinations of (3.11)–(3.13), we can make the exponent of $(z-\xi)^2+\epsilon^2 S(z)$ smaller by one in all of the kernels in (3.11)–(3.13). In particular, multiplying (3.11) by $\epsilon^2 S'(z)$, where the prime represents differentiation, and (3.13) by $2\epsilon^2 S(z)$ and adding, we obtain

$$\begin{aligned} &\epsilon^2 S'(z)\, B(\epsilon^2 S(z), z) + 2\epsilon^2 S(z)\, D(\epsilon^2 S(z), z) \\ &\quad = \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \left\{ \left[\frac{-2\epsilon^2 S(z)}{2n-1} \frac{d}{dz} \frac{1}{R^{n-\frac{1}{2}}} + \frac{2-n}{2n-1} \frac{\epsilon^2 S'(z)}{R^{n-\frac{1}{2}}} \right] \tilde{p}(\xi,\epsilon) \right. \\ &\qquad \left. - \frac{2\epsilon^2 S(z)}{2n+1} \frac{d}{dz} \frac{\tilde{b}(\xi,\epsilon)}{R^{n+\frac{1}{2}}} + \frac{2}{2n-1} \left[\frac{2(1-n)}{R^{n-\frac{1}{2}}} - \frac{d}{dz} \frac{z-\xi}{R^{n-\frac{1}{2}}} \right] \tilde{d}(\xi,\epsilon) \right\} d\xi. \end{aligned} \tag{4.4}$$

Multiplying (3.11) by 2 and (3.13) by $\epsilon^2 S'(z)$ and subtracting, we obtain

$$\epsilon^2 S'(z)\,D(\epsilon^2 S(z),z)-2B(\epsilon^2 S(z),z)=\int_{\alpha(\epsilon)}^{\beta(\epsilon)}\left\{\frac{2}{1-2n}\left[\frac{n}{R^{n-\frac12}}+\frac{d}{dz}\frac{z-\xi}{R^{n-\frac12}}\right]\tilde{p}(\xi,\epsilon)\right.$$
$$\left.-\frac{2}{2n+1}\left[\frac{2n}{R^{n+\frac12}}+\frac{d}{dz}\frac{z-\xi}{R^{n+\frac12}}\right]\tilde{b}(\xi,\epsilon)+\frac{2}{2n-1}\frac{d}{dz}\frac{1}{R^{n-\frac12}}\tilde{d}(\xi,\epsilon)\right\}d\xi. \qquad (4.5)$$

Finally, subtracting (3.11) from (3.12) and multiplying the resulting equation by $(S(z))^{-1}\{\epsilon^2(S'(z))^2+4S(z)\}$, we obtain

$$\{\epsilon^2(S'(z))^2+4S(z)\}\{C(\epsilon^2S(z),z)-B(\epsilon^2S(z),z)\}/S(z)$$
$$=\int_{\alpha(\epsilon)}^{\beta(\epsilon)}\left\{\frac{1}{1-2n}\left[\frac{8n-8}{R^{n-\frac12}}+\frac{4d}{dz}\frac{z-\xi}{R^{n-\frac12}}\right]-2\epsilon^2S'(z)\frac{d}{dz}\frac{1}{R^{n-\frac12}}\right]\tilde{p}(\xi,\epsilon)$$
$$\left.+\frac{1}{2n+1}\left[4\epsilon^2S'(z)\frac{d}{dz}\frac{1}{R^{n+\frac12}}-16n\frac{1}{R^{n+\frac12}}-\frac{8d}{dz}\frac{z-\xi}{R^{n+\frac12}}\right]\tilde{b}(\xi,\epsilon)\right\}d\xi. \qquad (4.6)$$

We shall now use (4.4)–(4.6) to solve for $\tilde{p}$, $\tilde{b}$, and $\tilde{d}$.

5. Asymptotic solution of the integral equations

We can now easily find the asymptotic expansion as $\epsilon\to 0$ of the solution to (4.4)–(4.6). Following the method of solution outlined at the beginning of §4, we employ the expansions of $I_k^j(z,\epsilon)$ given by

$$I_k^j(z,\epsilon)\equiv\int_{\alpha(\epsilon)}^{\beta(\epsilon)}\frac{(\xi-z)^j}{[(z-\xi)^2+\epsilon^2S(z)]^{k+\frac12}}(\xi-\alpha(\epsilon))^k(\beta(\epsilon)-\xi)^k f(\xi)\,d\xi$$
$$\sim\{\delta_{k,0}\delta_{j,1}+(1-\delta_{k,0}\delta_{j,1})\,\epsilon^{2j-2k}\}\sum_{r=0}^{\infty}\epsilon^{2r}(L_r^{k,j}+\log\epsilon^2G_r^{k,j})f(z) \qquad (5.1)$$

for $j=0,1,\quad k=0,1,2,\ldots$. Here $L_r^{k,j}$ and $G_r^{k,j}$ are certain linear operators which are defined in appendix B in terms of the operators $L_r^{0,j}$ and $G_r^{0,j}$ ($j=0,1$; $r\geqslant 0$) given by Handelsman & Keller (1967*a*, *b*).

We begin by defining new unknowns p, b and d by

$$\left.\begin{aligned}\tilde{p}(z,\epsilon)&=(z-\alpha(\epsilon))^{n-1}(\beta(\epsilon)-z)^{n-1}p(z,\epsilon),\\ \tilde{b}(z,\epsilon)&=(z-\alpha(\epsilon))^{n}(\beta(\epsilon)-z)^{n}b(z,\epsilon),\\ \tilde{d}(z,\epsilon)&=(z-\alpha(\epsilon))^{n-1}(\beta(\epsilon)-z)^{n-1}d(z,\epsilon).\end{aligned}\right\} \qquad (5.2)$$

The factors of the form $(z-\alpha(\epsilon))^k(\beta(\epsilon)-z)^k$ have been included in (5.2) because of the second requirement for the uniform expansion of the integrals I_k^j given by Geer (1975). In particular, it was shown that, in addition to the requirement on $\alpha(\epsilon)$ and $\beta(\epsilon)$ used in §4, it is both necessary and sufficient to require that $\tilde{f}$ in (4.1) vanishes to degree k at the end points of integration. That is, we must require that $\tilde{f}(z,\epsilon)=(z-\alpha)^k(\beta-z)^kf(z,\epsilon)$, where f is finite at α and β. Examination of (4.4)–(4.6) reveals that the value of k associated with $\tilde{p}$ and $\tilde{d}$ is $n-1$, while that associated with $\tilde{b}$ is n.

Since we have already determined α and β we see from (5.2) that $\tilde{p}$, $\tilde{b}$ and $\tilde{d}$ will be determined once the expansions for p, b and d have been found. However, as we shall see below, the form of the asymptotic expansion of these functions depends on whether $n = 1$ or $n > 1$, and so we consider these two cases separately.

For $n = 1$, the results (B 3) and (B 6) of appendix B show that the leading terms in the expansion of the integral operators on the right side of (4.4)–(4.6) are logarithmic in ϵ^2 [e.g. $2 \log \epsilon^2 p(z,\epsilon) - 2 \log \epsilon^2 d[d(z,\epsilon)]/dz$ on the right side of (4.5)]. This fact, coupled with the form of the expansions (5.1) and the fact that the left sides of (4.4)–(4.6) can be expanded in Taylor series in ϵ^2 about $\epsilon = 0$, suggests that we look for asymptotic expansions for p, b and d of the form

$$p(z,\epsilon) \sim \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{\epsilon^{2j}}{(\log \epsilon^2)^k} p_{j,k}(z), \tag{5.3a}$$

$$b(z,\epsilon) \sim \epsilon^2 \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{\epsilon^{2j}}{(\log \epsilon^2)^k} b_{j,k}(z), \tag{5.3b}$$

$$d(z,\epsilon) \sim \epsilon^2 \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{\epsilon^{2j}}{(\log \epsilon^2)^k} d_{j,k}(z). \tag{5.3c}$$

To determine the coefficients $p_{j,k}$, $b_{j,k}$ and $d_{j,k}$ in (5.3), we first expand each side of (4.4)–(4.6) with $n = 1$, using either Taylor's theorem or (5.1), and then insert the expansions (5.3) and collect coefficients of terms of the form $\epsilon^{2j}(\log \epsilon^2)^{-k}$. In this way, we are led, after some manipulation, to the following system of equations:

$$\begin{aligned} p_{j,1} = \tfrac{1}{2}g_j - \sum_{r=0}^{j-1} \Big[&\frac{d}{dz} G^{0,1}_{j-r} p_{r,1} - \tfrac{2}{3} G^{1,0}_{j-r} b_{r,1} - G^{0,0}_{j-r} p_{r,1} \\ &+ \frac{d}{dz}\left(\tfrac{1}{3} G^{1,1}_{j-1-r} b_{r,1} + G^{0,0}_{j-1-r} d_{r,1}\right)\Big], \quad j \geqslant 0, \end{aligned} \tag{5.4}$$

$$\begin{aligned} p_{j,k+1} = &\left\{\log \frac{4z(1-z)}{S(z)} + 2\right\} p_{j,k} + \int_0^1 \frac{p_{j,k}(\xi) - p_{j,k}(z)}{|\xi - z|}\, d\xi + \frac{4}{3}\frac{z(1-z)}{S(z)} b_{j,k} \\ &+ \sum_{r=0}^{j-1} \Big\{ L^{0,0}_{j-r} p_{r,k} + G^{0,0}_{j-r} p_{r,k+1} - \frac{d}{dz}\left(L^{0,1}_{j-r} p_{r,k} + G^{0,1}_{j-r} p_{r,k+1}\right) \\ &+ \tfrac{2}{3} L^{1,0}_{j-r} b_{r,k} + \tfrac{2}{3} G^{1,0}_{j-r} b_{r,k+1} - \frac{1}{3}\frac{d}{dz}\left(L^{1,1}_{j-1-r} b_{r,k} + G^{1,1}_{j-1-r} b_{r,k+1}\right) \\ &- \frac{d}{dz}\left(L^{0,0}_{j-1-r} d_{r,k} + G^{0,0}_{j-1-r} d_{r,k+1}\right)\Big\}, \quad j \geqslant 0, \quad k \geqslant 1, \end{aligned} \tag{5.5}$$

$$\begin{aligned} b_{j,k} = &-\frac{3}{4}\frac{S(z)}{z(1-z)} p_{j,k} + \frac{3}{16}\frac{S(z)}{z(1-z)} \sum_{r=0}^{j-1} \Big\{ 2\frac{d}{dz}\left(L^{0,1}_{j-r} p_{r,k} + G^{0,1}_{j-r} p_{r,k+1}\right) \\ &- \tfrac{8}{3}\left(L^{1,0}_{j-r} b_{r,k} + G^{1,0}_{j-r} b_{r,k+1}\right) - S'(z)\frac{d}{dz}\left(L^{0,0}_{j-1-r} p_{r,k} + G^{0,0}_{j-1-r} p_{r,k+1}\right) \\ &+ \tfrac{2}{3} S'(z)\frac{d}{dz}\left(L^{1,0}_{j-1-r} b_{r,k} + G^{1,0}_{j-1-r} b_{r,k+1}\right) \\ &+ \frac{4}{3}\frac{d}{dz}\left(L^{1,1}_{j-1-r} b_{r,k} + G^{1,1}_{j-1-r} b_{r,k+1}\right)\Big\}, \quad j \geqslant 0, \quad k \geqslant 1, \end{aligned} \tag{5.6}$$

$$d_{j,k} = \frac{1}{4}\sum_{r=0}^{j}\left\{S'(z)\,(L^{0,0}_{j-r}p_{r,k}+G^{0,0}_{j-r}p_{r,k+1}) - 2S(z)\frac{d}{dz}(L^{0,0}_{j-r}p_{r,k}+G^{0,0}_{j-r}p_{r,k+1}) - \tfrac{4}{3}S(z)\frac{d}{dz}(L^{1,0}_{j-r}b_{r,k}+G^{1,0}_{j-r}b_{r,k+1})\right\} + \frac{1}{2}\sum_{r=0}^{j-1}\frac{d}{dz}(L^{0,1}_{j-r}d_{r,k}+G^{0,1}_{j-r}d_{r,k+1}), \quad j \geqslant 0, \quad k \geqslant 1. \tag{5.7}$$

In (5.4), g_j is defined by

$$g_j(z) = \begin{cases} -2B(0,z) & \text{if } j = 0, \\ \dfrac{(S(z))^j}{j!}\left[-2\left(\dfrac{\partial}{\partial x}\right)^j B(x,z) + j\dfrac{S'(z)}{S(z)}\left(\dfrac{\partial}{\partial x}\right)^{j-1} D(x,z)\right]_{x=0} & \text{if } j \geqslant 1. \end{cases} \tag{5.8}$$

From (5.4)–(5.8) all of the coefficients $p_{j,k}$, $b_{j,k}$ and $d_{j,k}$ can be determined recursively. In particular, using (5.4)–(5.8) we find, for $j = 0$,

$$p_{0,1}(z) = -B(0,z), \tag{5.9a}$$

$$p_{0,k+1}(z) = \left\{\log\frac{4z(1-z)}{S(z)}+1\right\}p_{0,k}(z) + \int_0^1 \frac{p_{0,k}(\xi)-p_{0,k}(z)}{|\xi - z|}\,d\xi, \tag{5.9b}$$

$$b_{0,k}(z) = -\frac{3}{4}\frac{S(z)}{z(1-z)}p_{0,k}(z), \tag{5.9c}$$

$$d_{0,k}(z) = \tfrac{1}{2}S(z)p'_{0,k}(z) - \tfrac{1}{4}S'(z)p_{0,k}(z), \tag{5.9d}$$

for $k \geqslant 1$.

It is interesting to note that $C(r^2, z)$, which determines the θ component of the incident velocity field, does not appear in (5.4)–(5.8). Only $B(r^2,z)$ and $D(r^2,z)$ appear in (5.4) via the functions g_j. However, from the continuity equation, we see that C can be expressed explicitly in terms of B and D and hence all the essential information about the incident flow is contained in D and B.

For $n \geqslant 2$, the expressions (B 13) and (B 15) of appendix B indicate that the leading terms in the expansion of the right sides of (4.4)–(4.6) are now algebraically singular in ϵ^2 (e.g. the leading term involving $b(z,\epsilon)$ on the right side of (4.5) is $O(\epsilon^{-2n}b(z,\epsilon))$. Hence the form of the expansions (5.1) now suggest that we seek asymptotic expansions of p, b and d of the form

$$p(z,\epsilon) \sim \epsilon^{2n-2}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\epsilon^{2j}(\log\epsilon^2)^k p_{j,k}(z), \tag{5.10a}$$

$$b(z,\epsilon) \sim \epsilon^{2n}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\epsilon^{2j}(\log\epsilon^2)^k b_{j,k}(z), \tag{5.10b}$$

$$d(z,\epsilon) \sim \epsilon^{2n}\sum_{j=0}^{\infty}\sum_{k=0}^{j}\epsilon^{2j}(\log\epsilon^2)^k d_{j,k}(z). \tag{5.10c}$$

In a manner completely analogous to that described above, we substitute (5.10) into (4.4)–(4.6), perform the necessary expansions and collect coefficients of like terms to obtain the following set of relations:

$$p_{j,k}(z) = n\Gamma_{j,k}(z) + \frac{4(n-1)}{2n+1}\frac{z(1-z)}{S(z)}\Delta_{j,k}(z), \tag{5.11}$$

$$b_{j,k}(z) = -\Delta_{j,k}(z) - \frac{2n+1}{4}\frac{S(z)}{z(1-z)}\Gamma_{j,k}(z), \tag{5.12}$$

$$\begin{aligned} d_{j,k}(z) = \frac{2n-1}{4-4n}\frac{(2n-3)!}{[(n-2)!]^2} 2^{3-2n}\left(\frac{S(z)}{z(1-z)}\right)^{n-1}\Big\{\delta_{k,0} f_j(z) \\ + \sum_{r=k-1}^{j}\Big[\frac{2S(z)}{2n-1}\frac{d}{dz}(L^{n-1,0}_{j-r} p_{r,k} + G^{n-1,0}_{j-r} p_{r,k-1}) \\ + \frac{n-2}{2n-1} S'(z)(L^{n-1,0}_{j-r} p_{r,k} + G^{n-1,0}_{j-r} p_{r,k-1}) + \frac{2S(z)}{2n+1}\frac{d}{dz}(L^{n,0}_{j-r} b_{r,k} + G^{n,0}_{j-r} b_{r,k-1})\Big] \\ + \sum_{r=k-1}^{j-1}\Big[\frac{4n-4}{2n-1}(L^{n-1,0}_{j-r} d_{r,k} + G^{n-1,0}_{j-r} d_{r,k-1}) \\ - \frac{2}{2n-1}\frac{d}{dz}(L^{n-1,1}_{j-1-r} d_{r,k} + G^{n-1,1}_{j-1-r} d_{r,k-1})\Big]\Big\} \end{aligned} \tag{5.13}$$

for $j \geq 0$ and $0 \leq k \leq j$. The $\Gamma_{j,k}$ and $\Delta_{j,k}$ appearing in (5.11) and (5.12) are defined by

$$\begin{aligned} \Gamma_{j,k}(z) \equiv \frac{1-2n}{2n}\frac{(2n-3)!}{((n-2)!)^2} 2^{3-2n}\left(\frac{S(z)}{z(1-z)}\right)^{n-1}\Big\{\delta_{k,0} g_j(z) \\ + \sum_{r=k-1}^{j-1}\Big[\frac{2n}{2n-1}(L^{n-1,0}_{j-r} p_{r,k} + G^{n-1,0}_{j-r} p_{r,k-1}) + \frac{4n}{2n+1}(L^{n,0}_{j-r} b_{r,k} + G^{n,0}_{j-r} b_{r,k-1}) \\ - \frac{2}{2n-1}\frac{d}{dz}(L^{n-1,1}_{j-1-r} p_{r,k} + G^{n-1,1}_{j-1-r} p_{r,k-1}) \\ - \frac{2}{2n+1}\frac{d}{dz}(L^{n,1}_{j-1-r} b_{r,k} + G^{n,1}_{j-1-r} b_{r,k-1}) - \frac{2}{2n-1}\frac{d}{dz}(L^{n-1,0}_{j-1-r} d_{r,k} + G^{n-1,0}_{j-1-r} d_{r,k-1})\Big]\Big\}, \end{aligned} \tag{5.14}$$

$$\begin{aligned} \Delta_{j,k}(z) \equiv \frac{2n+1}{2^{2n+3}}\frac{(2n-1)!}{((n-1)!)^2}\left(\frac{S(z)}{z(1-z)}\right)^{n}\Big\{\delta_{k,0} h_{j-1}(z) \\ + \sum_{r=k-1}^{j-1}\Big[\frac{8n-8}{2n-1}(L^{n-1,0}_{j-r} p_{r,k} + G^{n-1,0}_{j-r} p_{r,k-1}) + \frac{16n}{2n+1}(L^{n,0}_{j-r} b_{r,k} + G^{n,0}_{j-r} b_{r,k-1}) \\ - \frac{4}{2n-1}\frac{d}{dz}(L^{n-1,1}_{j-1-r} p_{r,k} + G^{n-1,1}_{j-1-r} p_{r,k-1}) - \frac{2S'(z)}{2n-1}\frac{d}{dz}(L^{n-1,0}_{j-1-r} p_{r,k} + G^{n-1,0}_{j-1-r} p_{r,k-1}) \\ - \frac{4S'(z)}{2n+1}\frac{d}{dz}(L^{n,0}_{j-1-r} b_{r,k} + G^{n,0}_{j-1-r} b_{r,k-1}) - \frac{8}{2n+1}\frac{d}{dz}(L^{n,1}_{j-1-r} b_{r,k} + G^{n,1}_{j-1-r} b_{r,k-1})\Big]\Big\}. \end{aligned} \tag{5.15}$$

In (5.13)–(5.15), we define $b_{s,t} = d_{s,t} = p_{s,t} \equiv 0$ if $s < t$, $s < 0$ or $t < 0$. Also,

$$f_j(z) = \frac{(S(z))^j}{j!}\left[\left(\frac{\partial}{\partial x}\right)^j (S'(z)B(x,z) + 2S(z)D(x,z))\right]_{x=0} \quad \text{for} \quad j \geqslant 0, \tag{5.16}$$

$$h_{-1}(z) \equiv 0; \quad h_0(z) = 4S(z)\left[\frac{\partial}{\partial x}(C(x,z) - B(x,z))\right]_{x=0}, \tag{5.17a,b}$$

$$h_j(z) = 4\frac{(S(z))^{j+1}}{(j+1)!}\left[\left(\frac{\partial}{\partial x}\right)^{j+1}(C(x,z) - B(x,z))\right]_{x=0}$$
$$+(S'(z))^2\frac{(S(z))^{j-1}}{j!}\left[\left(\frac{\partial}{\partial x}\right)^j (C(x,z) - B(x,z))\right]_{x=0} \quad \text{for} \quad j \geqslant 1, \tag{5.17c}$$

while $g_j(z)$ is defined by (5.8).

From (5.11)–(5.17), all of the coefficients $p_{j,k}$, $b_{j,k}$ and $d_{j,k}$ can be determined recursively. In particular, setting $j = k = 0$ in (5.11)–(5.15) and using (5.16), (5.17) and (5.8), we find

$$p_{0,0}(z) = (2n-1)\frac{(2n-3)!}{[(n-2)!]^2}2^{3-2n}\left(\frac{S(z)}{z(1-z)}\right)^{n-1}B(0,z), \tag{5.18a}$$

$$b_{0,0}(z) = \left(\frac{1-4n^2}{n}\right)\frac{(2n-3)!}{[(n-2)!]^2}2^{1-2n}\left(\frac{S(z)}{z(1-z)}\right)^n B(0,z), \tag{5.18b}$$

$$d_{0,0}(z) = \frac{1-2n}{n-1}\frac{(2n-3)!}{[(n-2)!]^2}2^{1-2n}\left(\frac{S(z)}{z(1-z)}\right)^{n-1}\left\{(n-1)S'(z)B(0,z)\right.$$
$$\left.+\left(\frac{n+1}{n}\right)S(z)B_z(0,z) + 2S(z)D(0,z)\right\}. \tag{5.18c}$$

Equations (5.18) yield the leading terms in the asymptotic expansions of p, b and d in terms of the incident flow field.

6. Examples: uniform flow and shear flow

We now apply our results to the case of a uniform incident flow. Thus we set

$$\mathbf{v}^0 = U\cos\theta\mathbf{i}_r - U\sin\theta\mathbf{i}_\theta + W\mathbf{i}_z. \tag{6.1}$$

(Here $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_z$ are unit vectors in the directions of increasing r, θ and z respectively.) Using the notation of (3.5), we can set

$$\left.\begin{aligned} B_1(r^2,z) = C_1(r^2,z) = -U, \quad D_1(r^2,z) \equiv 0, \\ B_0(r^2,z) = C_0(r^2,z) \equiv 0, \quad D_0(r^2,z) = -W. \end{aligned}\right\} \tag{6.2}$$

We can then use (3.6)–(3.8) and (A 1)–(A 3) to represent $\mathbf{v}^b$ by formulae analogous to (3.9). In this way, we obtain the following representation for the velocity field $\mathbf{v} = \mathbf{v}^0 + \mathbf{v}^b$ and the pressure field p:

$$v_r = W\int_\alpha^\beta\left\{\frac{r(z-\xi)}{R^3}\tilde{p}(\xi,\epsilon) + \frac{r}{R^3}\tilde{b}(\xi,\epsilon)\right\}d\xi + U\cos\theta\left\{1 + \int_\alpha^\beta\left[\left(\frac{r^2}{R^3} + \frac{1}{R}\right)p(\xi,\epsilon)\right.\right.$$
$$\left.\left.+\frac{r^2}{R^5}(\xi-\alpha)(\beta-\xi)b(\xi,\epsilon) + \frac{(z-\xi)}{R^3}d(\xi,\epsilon)\right]d\xi\right\}, \tag{6.3}$$

$$v_\theta = -U\sin\theta\left\{1 + \int_\alpha^\beta\left[\frac{1}{R}p(\xi,\epsilon) - \frac{r^2}{R^5}(\xi-\alpha)(\beta-\xi)b(\xi,\epsilon) + \frac{(z-\xi)}{R^3}d(\xi,\epsilon)\right]d\xi\right\}, \tag{6.4}$$

$$v_z = W + W\int_\alpha^\beta \left\{\frac{2(z-\xi)^2+r^2}{R^3}\tilde{p}(\xi,\epsilon) + \frac{(z-\xi)}{R^3}\tilde{b}(\xi,\epsilon)\right\}d\xi$$

$$+ U\cos\theta\int_\alpha^\beta \left\{\frac{r(z-\xi)}{R^3}p(\xi,\epsilon) + \frac{r(z-\xi)(\xi-\alpha)(\beta-\xi)}{R^5}b(\xi,\epsilon) - \frac{r}{R^3}d(\xi,\epsilon)\right\}d\xi, \quad (6.5)$$

and

$$p = \int_\alpha^\beta \left\{\frac{2W(z-\xi)}{R^3}\tilde{p}(\xi,\epsilon) + U\cos\theta\frac{2r}{R^3}p(\xi,\epsilon)\right\}d\xi. \quad (6.6)$$

In (6.3)–(6.6), we have set $R = (r^2+(z-\xi)^2)^{\frac{1}{2}}$, while $\tilde{p}$, $\tilde{b}$, p, b and d represent the densities of the various singular flows. The leading terms in the asymptotic expansions of $\alpha = \alpha(\epsilon)$ and $\beta = \beta(\epsilon)$ are given by (4.2) and (4.3).

To determine the leading terms in the asymptotic expansions of $\tilde{p}$ and $\tilde{b}$, we use the results stated in appendix A. Thus we find, after some simplification,

$$\tilde{p}(z,\epsilon) \sim \sum_{k=1}^{\infty}(\log\epsilon^2)^{-k}\tilde{p}_{0,k}(z) + \epsilon^2\sum_{k=1}^{\infty}(\log\epsilon^2)^{-k}\tilde{p}_{1,k}(z) + O(\epsilon^4(\log\epsilon^2)^{-1}), \quad (6.7a)$$

$$\tilde{b}(z,\epsilon) \sim \epsilon^2\sum_{k=1}^{\infty}(\log\epsilon^2)^{-k}\tilde{b}_{0,k}(z) + O(\epsilon^4(\log\epsilon^2)^{-1}), \quad (6.7b)$$

where $\tilde{p}_{0,k}$, $\tilde{p}_{1,k}$ and $\tilde{b}_{0,k}$ are given by

$$\tilde{p}_{0,1} = \tfrac{1}{2}, \quad \tilde{p}_{0,2} = \tfrac{1}{2}\{\log(4z(1-z)/S(z)) - 1\}, \quad (6.8a)$$

$$\tilde{p}_{0,k+1} = \left\{\log\frac{4z(1-z)}{S(z)} - 1\right\}\tilde{p}_{0,k} + \int_0^1 \frac{\tilde{p}_{0,k}(\xi)-\tilde{p}_{0,k}(z)}{|\xi-z|}d\xi \quad \text{for} \quad k \geqslant 2, \quad (6.8b)$$

$$\tilde{p}_{1,1} = \tfrac{1}{8}S'(z), \quad (6.8c)$$

$$\begin{aligned}\tilde{p}_{1,k+1}(z) = &\left\{\log\frac{4z(1-z)}{S(z)} - 1\right\}\tilde{p}_{1,k} + \left(\int_0^{1-z} - \int_{-z}^0\right)\{\tilde{p}_{1,k}(z+v) - \tilde{p}_{1,k}(z)\}v^{-1}dv \\ &+ \frac{1}{4z}\left\{\frac{S(z)}{z}\tilde{p}_{0,k}(z) - S'(0)\tilde{p}_{0,k}(0)\right\} + \frac{1}{4(1-z)}\left\{\frac{S(z)}{1-z}\tilde{p}_{0,k}(z) + S'(1)\tilde{p}_{0,k}(1)\right\} \\ &- \frac{S(z)}{2}\frac{d}{dz}\left\{\tilde{p}'_{0,k}(z)\left[\log\frac{4z(1-z)}{S(z)} + 3\right]\right\} - \frac{1}{2}\frac{d}{dz}(S(z)\tilde{p}'_{0,k}(z)) \\ &+ \frac{S(z)}{4}\frac{d^2}{dz^2}\left\{\tilde{p}_{0,k}(z)\left[\log\frac{4z(1-z)}{S(z)} + 4\right]\right\} \\ &+ \frac{1}{4}\frac{d^2}{dz^2}\left\{S(z)\left(\int_0^{1-z} - \int_{-z}^0\right)\{\tilde{p}_{0,k}(z+v) - \tilde{p}_{0,k}(z)\}v^{-1}dv\right\} \\ &- \frac{1}{2}\left(\int_0^{1-z} - \int_{-z}^0\right)\{S(z+v)\tilde{p}_{0,k}(z+v) - S(z)\tilde{p}_{0,k}(z) - (S(z)\tilde{p}_{0,k}(z))'v \\ &- \tfrac{1}{2}(S(z)\tilde{p}_{0,k}(z))''v^2\}v^{-3}dv \quad \text{for} \quad k \geqslant 1, \end{aligned} \quad (6.8d)$$

$$\tilde{b}_{0,k} = \frac{1}{2}\frac{d}{dz}(S(z)\tilde{p}_{0,k}(z)) \quad \text{for} \quad k = 1, 2, \ldots. \quad (6.9)$$

In obtaining (6.8), we have used (6.2) to compute $u_0 = 2W$ and $u_1 = 0$.

When (6.8) and (6.9) are used in (6.7), they yield the asymptotic expansion of $\tilde{p}$ and $\tilde{b}$ up to terms which are $O(\epsilon^4(\log\epsilon^2)^{-1})$.

To determine the leading terms in the asymptotic expansions of p, b, and d, we use (5.3)–(5.8). In this way, we find, after some simplification,

$$p(z,\epsilon)\sim \sum_{k=1}^{\infty}(\log\epsilon^2)^{-k}p_{0,k}(z)+\epsilon^2\sum_{k=1}^{\infty}(\log\epsilon^2)^{-k}p_{1,k}(z)+O(\epsilon^4(\log\epsilon^2)^{-1}), \quad (6.10\,a)$$

$$b(z,\epsilon)\sim\epsilon^2\sum_{k=1}^{\infty}(\log\epsilon^2)^{-k}b_{0,k}(z)+O(\epsilon^4(\log\epsilon^2)^{-1}), \quad (6.10\,b)$$

$$d(z,\epsilon)\sim\epsilon^2\sum_{k=1}^{\infty}(\log\epsilon^2)^{-k}d_{0,k}(z)+O(\epsilon^4(\log\epsilon^2)^{-1}), \quad (6.10\,c)$$

where

$$p_{0,1}=1, \quad p_{0,2}(z)=\log(4z(1-z)/S(z))+1, \quad (6.11a)$$

$$p_{0,k+1}(z)=\left\{\log\frac{4z(1-z)}{S(z)}+1\right\}p_{0,k}(z)+\int_0^1\frac{p_{0,k}(\xi)-p_{0,k}(z)}{|\xi-z|}d\xi, \quad k\geqslant 2, \quad (6.11b)$$

$$p_{1,1}(z)=-\tfrac{1}{4}S''(z), \quad (6.11c)$$

$$\begin{aligned}p_{1,k+1}(z)=&\left\{\log\frac{4z(1-z)}{S(z)}+1\right\}p_{1,k}+\int_0^1\frac{p_{1,k}(\xi)-p_{1,k}(z)}{|\xi-z|}d\xi\\ &+\tfrac{1}{4}p_{0,k}\left\{\frac{(S'(z))^2}{S(z)}-6S''(z)-8\frac{S(z)}{z(1-z)}+3\frac{S'(z)(1-2z)}{z(1-z)}\right.\\ &\qquad\left.-\frac{4S(z)}{z^2(1-z)^2}(1-2z)^2\right\}\\ &+\tfrac{1}{4}p'_{0,k}\left\{3\frac{S(z)(1-2z)}{z(1-z)}+4S'(z)\right\}-\tfrac{1}{4}p''_{0,k}S(z)\left\{\log\frac{4z(1-z)}{S(z)}-5\right\}\\ &+\frac{3}{4}\frac{d^2}{dz^2}\left\{S(z)\int_0^1\frac{p_{0,k}(\xi)-p_{0,k}(z)}{|\xi-z|}d\xi\right\}\\ &\qquad-\frac{d}{dz}\left\{S'(z)\int_0^1\frac{p_{0,k}(\xi)-p_{0,k}(z)}{|\xi-z|}d\xi\right\}\\ &+\left(\int_0^{1-z}-\int_{-z}^0\right)\{G(z+v)-G(z)-G'(z)v\}v^{-2}dv\\ &+\frac{1}{4z}\left[\frac{S(z)}{z}p_{0,k}(z)-S'(0)p_{0,k}(0)\right]\\ &\qquad+\frac{1}{4(1-z)}\left[\frac{S(z)}{1-z}p_{0,k}(z)+S'(1)p_{0,k}(1)\right], \quad k\geqslant 1, \quad (6.11d)\end{aligned}$$

$$b_{0,1}(z)=-\frac{3S(z)}{4z(1-z)}, \quad b_{0,k}(z)=-\frac{3S(z)}{4z(1-z)}p_{0,k}(z) \quad \text{for} \quad k\geqslant 2, \quad (6.12)$$

$$d_{0,1}(z)=-\tfrac{1}{4}S'(z), \quad d_{0,k}(z)=\tfrac{3}{4}S(z)p'_{0,k}(z)-\tfrac{1}{4}S'(z)p_{0,k}(z) \quad \text{for} \quad k\geqslant 2. \quad (6.13)$$

In obtaining (6.11) we have used (6.2) to compute $g_0=2U$ and $g_1=0$. Also, in (6.11) we define $G(z)=\frac{1}{4}S'(z)p_{0,k}(z)-\frac{3}{4}S(z)p'_{0,k}(z)$.

When (6.11)–(6.13) and (6.8) and (6.9) are used in (6.10) and (6.7), respectively, and then (6.10) and (6.7) substituted into (6.3)–(6.6), these expressions yield the asymptotic expansions of the velocity and pressure fields around the body up to terms which are $O(\epsilon^4(\log\epsilon^2)^{-1})$.

As our second example, we consider a pure shear flow in which the flow is solely in a direction perpendicular to the axis of the body, say in the x direction, and varies linearly with y. Thus, we set

$$\mathbf{v}^0 = \tfrac{1}{2}r\sin 2\theta\,\mathbf{i}_r + \tfrac{1}{2}r(\cos 2\theta - 1)\,\mathbf{i}_\theta. \tag{6.14}$$

Again using the notation of (3.5) and (3.6) we can set

$$B_0(r^2,z) = D_0(r^2,z) = 0, \quad C_0(r^2,z) = \tfrac{1}{2}, \tag{6.15a}$$

and
$$B_2(r^2,z) = C_2(r^2,z) = -\tfrac{1}{2}, \quad D_2(r^2,z) = 0. \tag{6.15b}$$

Again, using the appropriate expressions from § 3 and appendix A, we are led to the following representation for $\mathbf{v}$ and p:

$$v_r = \sin 2\theta\left\{\tfrac{1}{2}r + \int_\alpha^\beta\left[\frac{r^3}{R^5}(\xi-\alpha)(\beta-\xi)p(\xi,\epsilon)\right.\right.$$
$$\left.\left. + \frac{r^3}{R^7}(\xi-\alpha)^2(\beta-\xi)^2 b(\xi,\epsilon) + \frac{r(z-\xi)}{R^5}(\xi-\alpha)(\beta-\xi)d(\xi,\epsilon)\right]d\xi\right\}, \tag{6.16}$$

$$v_\theta = \tfrac{1}{2}r(\cos 2\theta - 1) - \cos 2\theta\int_\alpha^\beta\left[\frac{r^3}{R^7}(\xi-\alpha)^2(\beta-\xi)^2 b(\xi,\epsilon)\right.$$
$$\left. - \frac{r(z-\xi)}{R^5}(\xi-\alpha)(\beta-\xi)d(\xi,\epsilon)\right]d\xi + \int_\alpha^\beta \frac{r}{R^3}(\xi-\alpha)(\beta-\xi)\tilde{d}(\xi,\epsilon)\,d\xi, \tag{6.17}$$

$$v_z = \sin 2\theta\int_\alpha^\beta\left[\frac{r^2(z-\xi)}{R^5}(\xi-\alpha)(\beta-\xi)p(\xi,\epsilon)\right.$$
$$\left. + \frac{r^2(z-\xi)}{R^7}(\xi-\alpha)^2(\beta-\xi)^2 b(\xi,\epsilon) - \frac{r^2}{R^5}(\xi-\alpha)(\beta-\xi)d(\xi,\epsilon)\right]d\xi, \tag{6.18}$$

$$p = \sin 2\theta\int_\alpha^\beta \frac{2r^2}{R^5}(\xi-\alpha)(\beta-\xi)p(\xi,\epsilon)\,d\xi. \tag{6.19}$$

In (6.16)–(6.19), $R = (r^2+(z-\xi)^2)^{\frac{1}{2}}$, while α and β are again given by (4.3) and (4.4).

To find the leading terms in the expansions (5.10) of p, b, d, and $\tilde{d}$, we use (5.10)–(5.17) with $n = 2$ and (A 10)–(A 11) to obtain, after some simplification,

$$p(z,\epsilon) \sim -3\frac{S(z)}{z(1-z)}\left\{(\tfrac{1}{2}\epsilon)^2 + (\tfrac{1}{2}\epsilon)^4\left[2S''(z)\right.\right.$$
$$\left. - \frac{1}{S(z)}\left\{(S'(z))^2 + [(S'(0)+S'(1))z - S'(0)]\frac{S(z)}{z(1-z)}\right\}\right]$$
$$\left. + (\tfrac{1}{2}\epsilon)^4\log\epsilon^2(S''(z) - S'(z))\right\} + O(\epsilon^6(\log\epsilon^2)^2), \tag{6.20}$$

$$b(z,\epsilon) \sim \tfrac{15}{32}(S(z)/z(1-z))^2\epsilon^4 + O(\epsilon^6\log\epsilon^2), \tag{6.21}$$

$$d(z,\epsilon) \sim \tfrac{3}{16}S'(z)\,(S(z)/z(1-z))\,\epsilon^4 + O(\epsilon^6\log\epsilon^2), \tag{6.22}$$

$$\tilde{d}(z,\epsilon) \sim \frac{S(z)}{z(1-z)}\Bigg\{(\tfrac{1}{2}\epsilon)^2 - \Bigg(\Bigg[\log\Bigg(\frac{4z(1-z)}{S(z)}\Bigg) - 1\Bigg] S''(z)$$

$$+ (1-2z)\frac{d}{dz}\Bigg(\frac{S(z)}{z(1-z)}\Bigg) + \Bigg[\frac{(1-2z)^2}{z(1-z)} + \frac{(S'(0)+S'(1))z - S'(0)}{S(z)}\Bigg]\frac{S(z)}{z(1-z)}$$

$$+ \Bigg(\int_0^{1-z} - \int_{-z}^{0}\Bigg)[S'(z+v) - S'(z) - S''(z)v]v^{-2}dv\Bigg)(\tfrac{1}{2}\epsilon)^4$$

$$+ S''(z)(\tfrac{1}{2}\epsilon)^4 \log\epsilon^2\Bigg\} + O(\epsilon^6(\log\epsilon^2)^2). \tag{6.23}$$

When (6.20)–(6.23) are used in (6.16)–(6.19), these expressions yield the asymptotic expansion of the velocity and pressure fields around the body up to terms which are $O(\epsilon^6(\log\epsilon^2)^2)$.

7. Force and torque on the body

We can now use our results to compute the total force $\tilde{\mathbf{F}}$ and torque $\tilde{\mathbf{N}}$ exerted on the body. For these purposes, it is convenient to use a Cartesian co-ordinate system (x, y, z) with the origin at one end of the body and the z axis coinciding with the axis of the body.

The momentum theorem for Stokes flow (i.e. neglecting the inertial terms) shows that

$$\tilde{\mathbf{F}} = \mu U a(\mathbf{F}^0 + \mathbf{F}^b), \tag{7.1}$$

where the Cartesian components of $\mathbf{F}^b$ are given by

$$F_x^b = \lim_{R\to\infty}\Bigg\{\int_{-R}^{R}\int_0^{2\pi}\Bigg[\cos\theta\Bigg[-p^b + 2\frac{\partial v_r^b}{\partial r}\Bigg] - \sin\theta\Bigg[\frac{\partial v_\theta^b}{\partial r} + \frac{1}{r}\frac{\partial v_r^b}{\partial\theta} - \frac{v_\theta^b}{r}\Bigg]\Bigg]_{\substack{r=R\\z=u}} R\,d\theta\,du$$

$$+ \int_0^{2\pi}\int_0^R\Bigg[\cos\theta\Bigg(\frac{\partial v_r^b}{\partial z} + \frac{\partial v_z^b}{\partial r}\Bigg) - \sin\theta\Bigg(\frac{\partial v_\theta^b}{\partial z} + \frac{1}{r}\frac{\partial v_z^b}{\partial\theta}\Bigg)\Bigg]_{r=u,\,z=-R}^{r=u,\,z=R} u\,du\,d\theta\Bigg\}, \tag{7.2}$$

$$F_z^b = \lim_{R\to\infty}\Bigg\{\int_{-R}^{R}\int_0^{2\pi}\Bigg[\frac{\partial v_z^b}{\partial r} + \frac{\partial v_r^b}{\partial z}\Bigg]_{r=R,\,z=u} R\,d\theta\,du$$

$$+ \int_0^{2\pi}\int_0^R\Bigg[\frac{2\partial v_z^b}{\partial z} - p^b\Bigg]_{r=u,\,z=-R}^{r=u\;\;z=R} u\,du\,d\theta\Bigg\}. \tag{7.3}$$

F_y^b is given by the right side of (7.2) with $\cos\theta$ replaced by $\sin\theta$ and $\sin\theta$ replaced by $-\cos\theta$. Similar expressions hold for the components of $\mathbf{F}^0$, with $\mathbf{v}^b$ replaced by $\mathbf{v}^0$ and p^b replaced by p^0. In (7.1), μ is the viscosity of the fluid, U is a typical velocity of the flow, and a is the (dimensional) length of the body. The formulae (7.1)–(7.3) were obtained by using as a control volume the volume of fluid bounded by the surface of the body and a large cylinder of length $2aR$ and radius Ra, which is 'centred' at the body and which has its axis coinciding with the z axis. Here R is a (large) dimensionless number. Also, in (7.2) and (7.3) we have used the notation $[G(r,z)]_{r=u,z=-R}^{r=u,z=R} = G(u,R) - G(u,-R)$.

Now the manner in which we 'Fourier analysed' both the incident and perturbed flow fields in § 3 and the formulae (7.2) and (7.3) allow us to make a useful observation. From (7.3) and the orthogonality of the cos $n\theta$ and sin $n\theta$ functions, it follows that the only component of $\mathbf{v}^b$ and p^b which can possibly contribute to F_z is the component corresponding to $n = 0$. Hence, using (A 1)–(A 3), (7.3) becomes

$$F_z^b = -8\pi \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \tilde{p}^0(\xi, \epsilon)\, d\xi. \tag{7.4}$$

In (7.4) we have placed a superscript zero on $\tilde{p}$ to remind us that this is the $\tilde{p}$ corresponding to $n = 0$. We can now substitute (5.3) in (7.4) and then expand the resulting integrals by Taylor's theorem to obtain the following complete expansion for F_z^b:

$$F_z^b = -8\pi \sum_{j=0}^{\infty} \epsilon^{2j} \sum_{k=1}^{\infty} (\log \epsilon^2)^{-k} \sum_{n=0}^{j} \frac{1}{n!} \left[\left(\frac{d}{d\epsilon^2} \right)^n \int_{\alpha(\epsilon)}^{\beta(\epsilon)} p^0_{j-n,\,k}(z)\, dz \right]_{\epsilon=0}. \tag{7.5}$$

In (7.5), the $p^0_{j,\,k}$ are determined by the formulae (A 7) and (A 8).

In a similar manner, the only component of $\mathbf{v}^b$ and p^b which can contribute to F_x or F_y is the component corresponding to $n = 1$. Hence, using (3.10) and the formulae analogous to (3.9) with $n = 1$, (7.3) yields

$$F_x^b = -8\pi \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \tilde{p}^1(\xi, \epsilon)\, d\xi, \quad F_y^b = 0, \tag{7.6}$$

if v_r^b is proportional to cos θ, and

$$F_x^b = 0, \quad F_y^b = -8\pi \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \tilde{p}^1(\xi, \epsilon)\, d\xi, \tag{7.7}$$

if v_r^b is proportional to sin θ. The superscript on p in (7.6) and (7.7) is to remind us that $\tilde{p}^1$ is the $\tilde{p}$-density corresponding to $n = 1$. The complete expansions for F_x^b and F_y^b can be found using (7.5) with $p^0_{j,\,k}$ replaced by $p^1_{j,\,k}$. Here the $p^1_{j,\,k}$ are determined by (5.4) and (5.5).

Thus, for any prescribed incident flow field, only the components of $\mathbf{v}^b$ and p^b corresponding to $n = 0$ and $n = 1$ need to be computed in order to determine the total force acting on the body, i.e. the components of $\mathbf{F}^b$ can be calculated from (7.4)–(7.7), while $\mathbf{F}^0$ can be computed from the given incident flow. We recognize the integrals in (7.4)–(7.7) as just the total strength of the Stokeslet distributions.

For our first example from the last section, i.e. a uniform incident flow, we can easily see that $\mathbf{F}^0 \equiv 0$. Thus, using formulae (6.7)–(6.8), (6.10) and (6.11) in (7.4) and (7.6), we find

$$\begin{aligned} \tilde{F}_x = 8\pi\mu U a \Bigg\{ & -(\log \epsilon^2)^{-1} - (\log \epsilon^2)^{-2} \left[1 + \int_0^1 \log \frac{4z(1-z)}{S(z)}\, dz \right] \\ & - (\log \epsilon^2)^{-3} \left[\int_0^1 \left(\left[\log \frac{4z(1-z)}{S(z)} + 1 \right]^2 \right. \right. \\ & \left. \left. + \int_0^1 |\xi - z|^{-1} \log \frac{\xi(1-\xi)\, S(z)}{S(\xi)\, z(1-z)}\, d\xi \right) dz \right] + O((\log \epsilon^2)^{-4}) \end{aligned}$$

$$+\epsilon^2(\log\epsilon^2)^{-2}\left[\frac{1}{4}\int_0^1\left\{S''(z)\log\left(\frac{4z(1-z)}{S(z)}\right)+\int_0^1\frac{S''(\xi)-S''(z)}{|\xi-z|}d\xi\right.\right.$$
$$+\frac{8S(z)}{z(1-z)}+\frac{4S(z)}{z^2(1-z)^2}(1-2z)^2-\frac{3S'(z)(1-2z)}{z(1-z)}-\frac{(S'(z))^2}{S(z)}+\frac{S'(1)}{4}[6+\log(-\tfrac{1}{4}S'(1))]$$
$$-\frac{S'(0)}{4}[6+\log\tfrac{1}{4}S'(0)]+\left(\int_0^{1-z}-\int_{-z}^0\right)[S'(z+v)-S'(z)-S''(z)v]v^{-2}dv\Big\}dz$$
$$\left.\left.-\int_0^1 z^{-2}(S(z)-S'(0)z)\,dz-\int_0^1(1-z)^{-2}(S(z)+(1-z)S'(1))\,dz\right\}\right]\Big\}+O(\epsilon^2(\log\epsilon^2)^{-3}), \tag{7.8}$$

$$\tilde{F}_y\equiv 0, \tag{7.9}$$

$$\tilde{F}_z=4\pi\mu Wa\left\{-(\log\epsilon^2)^{-1}-(\log\epsilon^2)^{-2}\left[1-\int_0^1\log\frac{4z(1-z)}{S(z)}dz\right]\right.$$
$$-(\log\epsilon^2)^{-3}\int_0^1\left(\left[\log\frac{4z(1-z)}{S(z)}-1\right]^2+\int_0^1|\xi-z|^{-1}\log\frac{\xi(1-\xi)S(z)}{S(\xi)z(1-z)}d\xi\right)dz$$
$$+O((\log\epsilon^2)^{-4})+\epsilon^2(\log\epsilon^2)^{-1}\tfrac{1}{4}(S'(0)-S'(1))$$
$$-\tfrac{1}{4}\epsilon^2(\log\epsilon^2)^{-2}\left[S'(0)-S'(1)+\int_0^1\left(S'(z)\log\frac{4z(1-z)}{S(z)}\right.\right.$$
$$+\left(\int_0^{1-z}-\int_{-z}^0\right)[S'(z+v)-S'(z)]v^{-1}dv$$
$$+\frac{1}{z}\left[\frac{S(z)}{z}-S'(0)\right]+\frac{1}{1-z}\left[\frac{S(z)}{1-z}+S'(1)\right]+S''(z)\log\frac{4z(1-z)}{S(z)}$$
$$\left.\left.-2\left(\int_0^{1-z}-\int_{-z}^0\right)[S(z+v)-S(z)-S'(z)v-S''(z)v^2/2]v^{-3}dv\right)dz\right]\Big\}$$
$$+O(\epsilon^2(\log\epsilon^2)^{-3}). \tag{7.10}$$

Equations (7.8)–(7.10) give some of the leading terms in the asymptotic expansion of the total force exerted upon the body by a uniform incident flow. Of course, more terms could be calculated using the recursion formulae in (6.8) and (6.11). The first two terms in each of these expansions were given by Tillett (1970). In comparing his formulae with ours, we must note that the length of his body is $2a$, whereas ours is just a. Of course, $\tilde{\mathbf{F}}\equiv 0$ for our second example.

We can also obtain formulae analogous to (7.2) and (7.3) for the total torque $\tilde{\mathbf{N}}$ exerted upon the body. For this purpose, it is convenient to introduce two new axes, the x_1 and x_2 axes, which are parallel to the x and y axes, respectively, but which lie in the plane $z=\frac{1}{2}$. We let $\tilde{N}_1$, $\tilde{N}_2$, and $\tilde{N}_z$ be the components of $\tilde{\mathbf{N}}$ about the x_1, x_2 and z axes, respectively. Then, by using a result similar to the momentum theorem involving moments about these three axes, we can write

$$\tilde{\mathbf{N}}=\mu Ua^2(\mathbf{N}^0+\mathbf{N}^b), \tag{7.11}$$

where the components of $\mathbf{N}^b$ are given by

$$N_1^b = \lim_{R\to\infty}\left\{\int_0^{2\pi}\int_0^R\left[\sin\theta\left\{2r\frac{\partial v_z^b}{\partial z} - rp - \left(z-\frac{1}{2}\right)\frac{\partial v_z^b}{\partial r} - \left(z-\frac{1}{2}\right)\frac{\partial v_r^b}{\partial z}\right\}\right.\right.$$
$$\left. - \left(z-\frac{1}{2}\right)\cos\theta\left\{\frac{\partial v_\theta^b}{\partial z} + \frac{\partial v_z^b}{\partial \theta}\right\}\right]_{z=-R}^{z=R} r\,dr\,d\theta$$
$$+\int_0^{2\pi}\int_{-R}^R\left[\sin\theta\left\{R\frac{\partial v_z^b}{\partial r} + R\frac{\partial v_r^b}{\partial z} + \left(z-\frac{1}{2}\right)p^b - (2z-1)\frac{\partial v_r^b}{\partial r}\right\}\right.$$
$$\left.\left. - \left(z-\frac{1}{2}\right)\cos\theta\left\{\frac{\partial v_\theta^b}{\partial r} - \frac{v_\theta^b}{r} + \frac{1}{r}\frac{\partial v_r^b}{\partial \theta}\right\}\right]_{r=R} R\,dz\,d\theta\right\}, \quad (7.12)$$

$$N_z^b = \lim_{R\to\infty}\left\{\int_0^{2\pi}\int_0^R\left[\left(\frac{\partial v_z^b}{\partial \theta} + r\frac{\partial v_\theta^b}{\partial z}\right)\right]_{z=-R}^{z=R} r\,dr\,d\theta\right.$$
$$\left.+\int_0^{2\pi}\int_{-R}^R\left[\left(\frac{\partial v_\theta^b}{\partial r} + \frac{1}{r}\frac{\partial v_r^b}{\partial \theta} - \frac{v_\theta^b}{r}\right)\right]_{r=R} R^2\,dz\,d\theta\right\}. \quad (7.13)$$

The expression for N_2^b is given by the right side of (7.12) with $\sin\theta$ replaced by $-\cos\theta$ and $\cos\theta$ replaced by $\sin\theta$. Similar expressions hold for the components of $\mathbf{N}^0$, with $\mathbf{v}^b$ replaced by $\mathbf{v}^0$ and p^b replaced by p^0.

From (7.13) it follows that the only component of $\mathbf{v}^b$ and p^b which can contribute to N_z^b corresponds to $n = 0$. Hence, using (A 1)–(A 3), (7.13) becomes

$$N_z^b = -8\pi\int_{\alpha(\epsilon)}^{\beta(\epsilon)} \tilde{d}^0(\xi,\epsilon)\,d\xi, \quad (7.14)$$

where $\tilde{d}^0$ denotes the $\tilde{d}$-density corresponding to $n = 0$ which appears in (A 3). Substituting $\tilde{d}^0 = (\beta-\xi)(\xi-\alpha)\,d^0$, where d^0 has an expansion of the form (5.10) with $n = 1$, in (7.14), we can write

$$N_z^b = -8\pi\epsilon^2\sum_{j=0}^{\infty}\sum_{k=0}^{j}\epsilon^{2j}(\log\epsilon^2)^k\sum_{n=k}^{j}\frac{1}{(j-n)!}$$
$$\times\left[\left(\frac{d}{d\epsilon^2}\right)^{j-n}\int_{\alpha(\epsilon)}^{\beta(\epsilon)}(z-\alpha(\epsilon))\,(\beta(\epsilon)-z)\,d_{n,k}^0(z)\,dz\right]_{\epsilon=0}, \quad (7.15)$$

which gives us a complete expansion for N_z^b. In (7.15), the $d_{j,k}^0$ are determined recursively from (A 10).

In a similar manner, we see that the only component of the perturbed flow field which can contribute to N_1^b and N_2^b corresponds to $n = 1$. Hence (7.12) yields

$$N_1^b = 8\pi\left\{\int_{\alpha(\epsilon)}^{\beta(\epsilon)}\left(\xi-\frac{1}{2}\right)\tilde{p}^1(\xi,\epsilon)\,d\xi + \int_{\alpha(\epsilon)}^{\beta(\epsilon)}\tilde{d}^1(\xi,\epsilon)\,d\xi\right\}, \quad (7.16)$$

and $N_2^b = 0$, if v_r is proportional to $\sin\theta$. Also $N_1 = 0$ and N_2 is given by the negative of (7.16) if v_r is proportional to $\cos\theta$. Again, the superscript on p and d indicates that these densities correspond to $n = 1$. The complete expansion of N_1^b can be found by using (7.5) with $p_{j,k}^0$ replaced by $(z-\frac{1}{2})p_{j,k}^1(z) + d_{j-1,k}^1$. Here $p_{j,k}^1$ and $d_{j,k}^1$ are determined by (5.4)–(5.7). Equations (7.14)–(7.16) give us the components of $\mathbf{N}^b$ in terms of the densities calculated in § 5. $\mathbf{N}^0$ can be computed from the given incident flow field. In particular, from (7.15), we see that N_z^b is $O(\epsilon^2)$ and its leading term is given by

$$N_z^b = -4\pi\epsilon^2 \int_0^1 S(\xi)\, V(0,\xi)\, d\xi + O(\epsilon^4 \log \epsilon^2), \tag{7.17}$$

where

$$V(0,z) = \frac{1}{2\pi}\int_0^{2\pi} v_\theta^0(0,\theta,z)\, d\theta.$$

Also, the first integral in (7.16) is just the first moment of the Stokeslet distribution corresponding to $n = 1$. From (5.3) we see that this integral is only $O((\log \epsilon^2)^{-1})$, while the second integral is $O(\epsilon^2(\log \epsilon^2)^{-1})$. Thus, using (5.9), (7.15) yields

$$N_1^b = -8\pi(\log \epsilon^2)^{-1}\int_0^1 (\xi - \tfrac{1}{2})\, B(0,\xi)\, d\xi + O((\log \epsilon^2)^{-2}), \tag{7.18}$$

where

$$B(0,z) = -\frac{1}{\pi}\int_0^{2\pi} v_r^0(0,\theta,z)\sin\theta\, d\theta.$$

In our first example of § 6, we see that $\mathbf{N}^0 \equiv 0$ and also that $N_1^b = N_z^b = 0$. Now using (6.11) and (6.13) in (6.10), (7.16) yields

$$\begin{aligned}
\tilde{N}_2 = {} & -8\pi\mu U a^2\Bigg\{(\log\epsilon^2)^{-2}\int_0^1 \left(z-\frac{1}{2}\right)\log\frac{4z(1-z)}{S(z)}\,dz \\
& + (\log\epsilon^2)^{-3}\int_0^1\Bigg(\left[\log\frac{4z(1-z)}{S(z)}+1\right]^2 \\
& \qquad + \int_0^1 |\xi - z|^{-1}\log\frac{\xi(1-\xi)\,S(z)}{z(1-z)\,S(\xi)}\,d\xi\Bigg)dz + O((\log\epsilon^2)^{-4}) \\
& + \epsilon^2(\log\epsilon^2)^{-2}\Bigg(\int_0^1\Bigg[\frac{(1-2z)\,S(z)}{z(1-z)} - \tfrac{1}{8}(2z-1)\Bigg\{S''(z)\left[\log\frac{4z(1-z)}{S(z)}+1\right] \\
& \qquad + \int_0^1\frac{S''(\xi)-S''(z)}{|\xi - z|}\,d\xi + 6S''(z) + \frac{4S(z)}{z^2(1-z)^2}(1-2z)^2 \\
& \quad - \frac{3S'(z)\,(1-2z)}{z(1-z)} - \frac{(S'(z))^2}{S(z)} + \left(\int_0^{1-z} - \int_{-z}^0\right)[S'(z+v) - S'(z) - S''(z)\,v]\,v^{-2}\,dv \\
& \qquad - \frac{1}{z}\left(\frac{S(z)}{z} - S'(0)\right) - \frac{1}{1-z}\left(\frac{S(z)}{1-z} + S'(1)\right)\Bigg\}\Bigg]\,dz \\
& + \tfrac{1}{8}S'(1)\,(1-\log(-\tfrac{1}{4}S'(1))) + \tfrac{1}{8}S'(0)\,(1-\log\tfrac{1}{4}S'(0))\Bigg)\Bigg\} + O(\epsilon^2(\log\epsilon^2)^{-3}).
\end{aligned} \tag{7.19}$$

In our second example of § 6, we see again that $\mathbf{N}^0 \equiv 0$, while $N_1^b = N_2^b = 0$. Hence, using (6.23) in (7.15), (7.11) yields

$$\begin{aligned}
\tilde{N}_z = {} & -8\pi\mu U a^2\Bigg\{\left(\frac{\epsilon}{2}\right)^2\int_0^1 S(z)\,dz - \left(\frac{\epsilon}{2}\right)^4\log\epsilon^2\int_0^1 (S'(z))^2\,dz \\
& - \left(\frac{\epsilon}{2}\right)^4\int_0^1 S(z)\Bigg\{\left[\log\frac{4z(1-z)}{S(z)} - 1\right]S''(z) + (1-2z)\frac{d}{dz}\left(\frac{S(z)}{z(1-z)}\right) \\
& \qquad + \left(\int_0^{1-z} - \int_{-z}^0\right)[S'(z+v) - S'(z) - S''(z)\,v]\,v^{-2}\,dv
\end{aligned}$$

$$+\left[\frac{(1-2z)^2}{z(1-z)}+\frac{(S'(0)+S'(1))z-S'(0)}{S(z)}\right]\frac{S(z)}{z(1-z)}$$
$$+[S'(0)-(S'(1)+S'(0))z]\frac{S(z)}{z(1-z)}\bigg\}dz\bigg\}+O(\epsilon^6(\log\epsilon^2)^2). \quad (7.20)$$

As before, more terms can be computed in (7.18) and (7.19) by using the recursion formulae in (6.11) and (6.13).

This research was supported in part by the Research Foundation of the State University of New York.

Appendix A

In this appendix we state the results, for the case of $n = 0$, which correspond to the results in §§3–5 above. In particular, the singular solutions for $n = 0$, corresponding to (3.6)–(3.8), can be found easily and lead to the following representation for $\mathbf{v}^b$ and p^b:

$$v_r^b(r,z,\epsilon) = \int_\alpha^\beta \left\{\frac{(z-\xi)\,r}{R^3}\tilde{p}(\xi,\epsilon)+\frac{r}{R^3}\tilde{b}(\xi,\epsilon)\right\}d\xi, \quad (\text{A }1)$$

$$v_z^b(r,z,\epsilon) = \int_\alpha^\beta \left\{\frac{2(z-\xi)^2+r^2}{R^3}\tilde{p}(\xi,\epsilon)+\frac{(z-\xi)}{R^3}\tilde{b}(\xi,\epsilon)\right\}d\xi, \quad (\text{A }2)$$

$$v_\theta^b(r,z,\epsilon) = \int_\alpha^\beta \frac{r}{R^3}\tilde{d}(\xi,\epsilon)\,d\xi, \quad p^b = \int_\alpha^\beta \frac{2(z-\xi)}{R^3}\tilde{p}(\xi,\epsilon)\,d\xi. \quad (\text{A }3)$$

The singular solutions used in (A 1) and (A 2) correspond to a Stokeslet and to an irrotational source. These are the singular solutions used by Tillett (1970).

We now use (3.5) with $n = 0$ in (3.1) to obtain equations analogous to (3.11)–(3.13). We can then rewrite these equations, as in §4, to obtain the equations

$$\epsilon^2S'(z)\,W(\epsilon^2S(z),z)-2\epsilon^2S(z)\,U(\epsilon^2S(z),z) = \int_{\alpha(\epsilon)}^{\beta(\epsilon)}\left\{\frac{2d}{dz}\frac{\epsilon^2S(z)}{R^{\frac{1}{2}}}\tilde{p}(\xi,\epsilon) - \frac{2d}{dz}\frac{z-\xi}{R^{\frac{1}{2}}}\tilde{b}(\xi,\epsilon)\right\}d\xi, \quad (\text{A }4)$$

$$V(\epsilon^2S(z),z) = \int_{\alpha(\epsilon)}^{\beta(\epsilon)} R^{-\frac{1}{2}}\,\tilde{d}(\xi,\epsilon)\,d\xi, \quad (\text{A }5)$$

$$\epsilon^2S'(z)\,U(\epsilon^2S(z),z)+2W(\epsilon^2S(z),z) = \int_{\alpha(\epsilon)}^{\beta(\epsilon)}\left\{\left[\frac{4}{R^{\frac{1}{2}}}-2\frac{d}{dz}\frac{z-\xi}{R^{\frac{1}{2}}}\right]\tilde{p}(\xi,\epsilon) - 2\frac{d}{dz}\frac{1}{R^{\frac{1}{2}}}\tilde{b}(\xi,\epsilon)\right\}d\xi. \quad (\text{A }6)$$

In (A 4)–(A 6) we have used the notation $U = r^{-2}B_0$, $V = ir^{-2}C_0$, $W = D_0$ and $R = \epsilon^2S(z) + (z-\xi)^2$.

We now look for asymptotic expansions for $\tilde{p}$ and $\tilde{b}$ in the form of the right side of (5.3), while we write $\tilde{d} = (z-\alpha)\,(\beta-z)\,d$ and look for an expansion of d in the form of (5.10) with $n = 1$. Thus, using the procedure outlined in §§4 and 5, we are led to the following system of relations:

$$p_{j,1}(z) = -\frac{1}{4}\left\{u_j(z) - \sum_{r=0}^{j-1}\left(4G^{0,0}_{j-r}p_{r,1} + 2\frac{d}{dz}G^{0,1}_{j-r}p_{r,1} - 2\frac{d}{dz}G^{0,0}_{j-r-1}b_{r,1}\right)\right\} \quad \text{for} \quad j \geqslant 0, \tag{A 7}$$

$$p_{j,k+1}(z) = L^{0,0}_0 p_{j,k} - p_{j,k} + \sum_{r=0}^{j-1}\left\{L^{0,0}_{j-r}p_{r,k} + G^{0,0}_{j-r}p_{r,k+1}\right.$$
$$\left. + \frac{1}{2}\frac{d}{dz}(L^{0,1}_{j-r}p_{r,k} + G^{0,1}_{j-r}p_{r,k+1} - L^{0,0}_{j-1-r}b_{r,k} - G^{0,0}_{j-1-r}b_{r,k+1})\right\} \quad \text{for} \quad k \geqslant 1, \quad j \geqslant 0, \tag{A 8}$$

$$b_{j,k}(z) = \frac{1}{2}\frac{d}{dz}\left\{S(z)L^{0,0}_0 p_{j,k} - S(z)p_{j,k+1} + \sum_{r=0}^{j-1}[S(z)L^{0,0}_{j-r}p_{r,k} + S(z)G^{0,0}_{j-r}p_{r,k+1}\right.$$
$$\left. + L^{0,1}_{j-r}b_{r,k} + G^{0,1}_{j-r}b_{r,k+1}]\right\} \quad \text{for} \quad j \geqslant 0, \quad k \geqslant 1, \tag{A 9}$$

$$d_{j,k}(z) = \frac{S(z)}{2z(1-z)}\left\{v_j(z)\,\delta_{k,0} - \sum_{r=k-1}^{j-1}(L^{1,0}_{j-r}d_{r,k} + G^{1,0}_{j-r}d_{r,k-1})\right\}$$
$$\text{for} \quad j \geqslant 0, \quad 0 \leqslant k \leqslant j. \tag{A 10}$$

(Here we define $d_{s,t} \equiv 0$ if $s < t$, $s < 0$, or $t < 0$.) In (A 7)–(A 10), u_j and v_j are defined by

$$u_j(z) = \begin{cases} 2W(0,z) & \text{if} \quad j = 0, \\ \dfrac{(S(z))^{j-1}}{(j-1)!}\left[\dfrac{2S(z)}{j}\dfrac{\partial^j W}{\partial x^j}(x,z) + S'(z)\dfrac{\partial^{j-1}U(x,z)}{\partial x^{j-1}}\right]_{x=0} & \text{if} \quad j \geqslant 1; \end{cases}$$

$$v_j(z) = \frac{(S(z))^j}{j!}\left[\frac{\partial^j V(x,z)}{\partial x^j}\right]_{x=0} \quad \text{if} \quad j \geqslant 0. \tag{A 11}$$

From (A 7)–(A 10) the coefficients $p_{j,k}$, $b_{j,k}$, and $d_{j,k}$ can be determined recursively. In particular, from (A 8) and (A 9) it follows that

$$b_{0,k}(z) = \frac{d}{dz}(\tfrac{1}{2}S(z)\,p_{0,k}(z)), \quad k \geqslant 1, \tag{A 12}$$

while the $p_{0,k}$ are determined from (A 7) and (A 8) by

$$p_{0,1}(z) = -\tfrac{1}{2}W(0,z),$$
$$p_{0,k+1}(z) = \left\{\log\left(\frac{4z(1-z)}{S(z)}\right) - 1\right\}p_{0,k}(z) + \int_0^1 \frac{p_{0,k}(\xi) - p_{0\,k}(z)}{|\xi - z|}\,d\xi, \tag{A 13}$$

while from (A 10) we find

$$d_{0,0}(z) = \frac{S(z)}{2z(1-z)}V(0,z). \tag{A 14}$$

In obtaining (A 13) we have used the expression for the operator $L^{0,0}_0$ given by (B 7) in appendix B.

The result (A 12) is equivalent to a result given by Tillett (1970), while (A 13) reduces to his results if we set $W(r^2, z) = W$ (a constant).

Appendix B. The operators $L_r^{k,j}$ and $G_r^{k,}$

In this appendix, we present an explicit recurrence formula for the linear operators $L_r^{k,j}$ and $G_r^{k,j}$, which appear in the expansion of $I_k^j(z,\epsilon)$ defined by (5.1). More precisely, they are are coefficients of ϵ^{2r} and $\epsilon^{2r}\log\epsilon^2$ in the expansion of

$$I_k^j(z,\epsilon,F) \equiv \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{(\xi-z)^j}{[(\xi-z)^2+\epsilon^2 S(z)]^{k+\frac{1}{2}}}(\xi-\alpha(\epsilon))^k(\beta(\epsilon)-\xi)^k F(\xi)\,d\xi$$

$$\sim \begin{cases} \sum_{r=0}^{\infty} \epsilon^{2r}(L_r^{0,1}+\log\epsilon^2 G_r^{0,1})F(z) & \text{if} \quad k=0 \quad \text{and} \quad j=1, \\ \epsilon^{2j-2k}\sum_{r=0}^{\infty}\epsilon^{2r}(L_r^{k,j}+\log\epsilon^2 G_r^{k,j})F(z) & \text{if} \quad k=0 \quad \text{and} \quad j=0 \\ & \text{or} \quad k\geqslant 1 \quad \text{and} \quad j=0,1. \end{cases} \tag{B 1}$$

Here we assume that $\alpha(\epsilon)$ and $\beta(\epsilon)$ have expansions of the form

$$\alpha(\epsilon) \sim \sum_{n=0}^{\infty}\alpha_n\epsilon^{2n}, \quad \beta(\epsilon)\sim\sum_{n=0}^{\infty}\beta_n\epsilon^{2n}, \tag{B 2}$$

where the coefficients α_n and β_n can be determined as described in §4.

We will now show how we can express $L_r^{k,j}$ in terms of the operators $L_p^{n,m}$, with $n<k$. (A similar result will hold for the operators $G_r^{k,j}$.) We begin by noting that, for $k=0$, the operators $L_r^{0,1}$ and $G_r^{0,1}$ have been defined by Handelsman & Keller (1967*a*), while $L_r^{0,0}$ and $G_r^{0,0}$ have been defined in Handelsman & Keller (1967*b*). In particular, we find

$$L_0^{0,1}F(z) = \int_z^1 F(\xi)\,d\xi - \int_0^z F(\xi)\,d\xi, \quad G_0^{0,1}F(z)\equiv 0, \tag{B 3}$$

$$L_1^{0,1}F(z) = \tfrac{1}{4}F(1)S'(1)+\tfrac{1}{4}F(0)S'(0)-\frac{1}{2}\left\{F'(z)S(z)\left[\log\frac{4z(1-z)}{S(z)}-1\right]\right.$$
$$\left.+F(z)\left[\frac{S(z)}{z}-\frac{S(z)}{1-z}\right]+S(z)\left(\int_0^{1-z}-\int_{-z}^0\right)\{F(z+v)-F(z)-F'(z)v\}v^{-2}dv\right\}, \tag{B 4}$$

$$G_1^{0,1}F(z) = \tfrac{1}{2}S(z)F'(z), \quad G_2^{0,1}F(z) = -\tfrac{1}{16}S^2(z)F'''(z) \tag{B 5}$$

and

$$G_0^{0,0}F(z) = -F(z), \quad G_1^{0,0}F(z) = \tfrac{1}{4}S(z)F''(z), \tag{B 6}$$

$$L_0^{0,0}F(z) = F(z)\log\frac{4z(1-z)}{S(z)}+\int_0^1\frac{F(\xi)-F(z)}{|\xi-z|}d\xi,$$

$$L_1^{0,0}F(z) = \frac{1}{4z}\left\{\frac{S(z)}{z}F(z)-S'(0)F(0)\right\}+\frac{1}{4(1-z)}\left\{\frac{S(z)F(z)}{1-z}+S'(1)F(1)\right\}$$
$$+\frac{S(z)(2z-1)}{2z(1-z)}F'(z)+\tfrac{1}{2}S(z)\left\{1-\log\frac{4z(1-z)}{S(z)}\right\}F''(z)$$
$$-\tfrac{1}{2}S(z)\left(\int_0^{1-z}-\int_{-z}^0\right)\{F(z+v)-F(z)-F'(z)v-\tfrac{1}{2}F''(z)v^2\}v^{-3}dv. \tag{B 7}$$

Now, for $k \geqslant 0$, we can write I^1_{k+1} in the form

$$I^1_{k+1}(z,\epsilon;F) = -\frac{1}{2k+1}\int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{d}{d\xi}\left\{\frac{1}{R^{k+\frac{1}{2}}}\right\}(\xi-\alpha(\epsilon))^{k+1}(\beta(\epsilon)-\xi)^{k+1}F(\xi)\,d\xi$$

$$= \frac{1}{2k+1}\int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{1}{R^{k+\frac{1}{2}}}\frac{d}{d\xi}\{(\xi-\alpha(\epsilon))^{k+1}(\beta(\epsilon)-\xi)^{k+1}F(\xi)\}\,d\xi$$

$$= \frac{1}{2k+1}I^0_k(z,\epsilon;(\xi-\alpha(\epsilon))(\beta(\epsilon)-\xi)F'(\xi)+(k+1)(\beta(\epsilon)+\alpha(\epsilon)-2\xi)F(\xi)), \quad \text{(B 8)}$$

where we have used integration by parts in obtaining (B 8). Using the expansions (B 1) and (B 2) in (B 8) and comparing coefficients of terms of the form $\epsilon^{2r}(\log \epsilon^2)^n$ we obtain

$$(2k+1)L^{k+1,1}_r F(z) = -L^{k,0}_r(z^2F'(z)+2(k+1)zF(z)) + \sum_{j=0}^{r}\left\{(\alpha_{r-j}+\beta_{r-j})L^{k,0}_j(zF'(z)+(k+1)F(z)) - \sum_{i=0}^{r-j}\alpha_i\beta_{r-j-i}L^{k,0}_j(F'(z))\right\}. \quad \text{(B 9)}$$

The operators $(2k+1)G^{k+1,1}_r$ are given by the right side of (B 9) with $L^{k,0}_j$ replaced by $G^{k,0}_j$.

Also, we can write I^0_{k+1} as

$$I^0_{k+1}(z,\epsilon;F) = \frac{1}{\epsilon^2 S(z)}\int_{\alpha(\epsilon)}^{\beta(\epsilon)}\left\{\frac{1}{R^{k+\frac{1}{2}}}+\frac{\xi-z}{2k+1}\frac{d}{d\xi}\frac{1}{R^{k+\frac{1}{2}}}\right\}(\xi-\alpha(\epsilon))^{k+1}(\beta(\epsilon)-\xi)^{k+1}F(\xi)\,d\xi \quad \text{(B 10)}$$

for $k \geqslant 0$. Integrating the last term by parts and then simplifying the result, (B 10) becomes

$$I^0_{k+1}(z,\epsilon;F) = [(2k+1)\epsilon^2 S(z)]^{-1}\{I^0_k(z,\epsilon;2k(z-\alpha(\epsilon))(\beta(\epsilon)-z)F(z)) - I^1_k(z,\epsilon;(z-\alpha(\epsilon))(\beta(\epsilon)-z)F'(z)+(k+1)[\alpha(\epsilon)+\beta(\epsilon)-2z]F(z))\}. \quad \text{(B 11)}$$

Again using the expansions (B 1) and (B 2) in (B 11), we are led to the relation

$$(2k+1)S(z)L^{k+1,0}_r F(z) = 2k\left\{-L^{k,0}_r(z^2F(z)) + \sum_{j=0}^{r}\left[(\beta_{r-j}+\alpha_{r-j})L^{k,0}_j(zF(z)) - \sum_{i=0}^{r-j}\alpha_i\beta_{r-j-i}L^{k,0}_j F(z)\right]\right\}$$
$$+\begin{cases} L^{k,1}_{r-1}(z^2F'(z)+2(k+1)zF(z)) - \sum_{j=0}^{r-1}\left[(\alpha_{r-1-j}+\beta_{r-1-j})L^{k,1}_j((k+1)F(z)+zF'(z)) - \sum_{i=0}^{r-1-j}\alpha_i\beta_{r-1-j-i}L^{k,1}_j(F'(z))\right] & \text{if } k \geqslant 1, \\ L^{0,1}_r(z^2F'(z)+2zF(z)) - \sum_{j=0}^{r}\left[(\alpha_{r-j}+\beta_{r-j})L^{0,1}_j(zF'(z)+F(z)) - \sum_{i=0}^{r-j}\alpha_i\beta_{r-j-i}L^{0,1}_j(F'(z))\right] & \text{if } k = 0. \end{cases} \quad \text{(B 12)}$$

In (B 12), we define $L_{-1}^{k,1} \equiv 0$. The operators $(2k+1)\,S(z)\,G_r^{k+1,0}F(z)$ are given by the right side of (B 12) with $L_r^{k,j}$ replaced by $G_r^{k,j}$. Equations (B 9) and (B 12) are the desired recurrence relations. They express $L_r^{k+1,j}$ and $G_r^{k+1,j}$ in terms of $L_t^{k,s}$ and $G_t^{k,s}$. Thus these relations and the operators defined by Handelsman & Keller serve to define all of our operators $L_r^{k,j}$ and $G_r^{k,j}$.

In particular, we can now use (B 9) and (B 12) with (B 3)–(B 7) to calculate a few of these operators. In this way we find

$$G_0^{1,1}F(z) = d[z(1-z)\,F(z)]/dz, \quad G_0^{k,1}F(z) \equiv 0 \quad \text{for} \quad k \geqslant 2, \tag{B 13a}$$

$$G_0^{k,0}F(z) \equiv 0 \quad \text{for} \quad k \geqslant 1, \tag{B 13b}$$

$$G_1^{1,0}F(z) = -\tfrac{1}{2}d^2[z(1-z)\,F(z)]/dz^2, \quad G_1^{2,0}F(z) \equiv 0, \tag{B 13c}$$

and $$L_0^{1,0}F(z) = 2z(1-z)\,F(z)/S(z), \tag{B 14a}$$

$$L_1^{1,0}F(z) = \frac{1}{2}\Bigg\{\left[\log\frac{4z(1-z)}{S(z)} - 1\right]\frac{d^2}{dz^2}[z(1-z)\,F(z)] + (1-2z)\,F'(z)$$
$$\left(\int_0^{1-z} - \int_{-z}^{0}\right)[H(z+v) - H(z) - H'(z)\,v]\,v^{-2}\,dv$$
$$+\left[\frac{(1-2z)^2}{z(1-z)} + \frac{(S'(0)+S'(1))\,z - S'(0)}{S(z)}\right]F(z)\Bigg\}, \tag{B 14b}$$

$$L_0^{1,1}F(z) = \log\frac{4z(1-z)}{S(z)}\frac{d}{dz}[z(1-z)\,F(z)]$$
$$+\int_0^1 \frac{1}{|\xi - z|}\left[\frac{d}{dx}(x(1-x)\,F(x))\right]_{x=z}^{x=\xi} d\xi, \tag{B 14c}$$

$$L_0^{2,0}F(z) = \tfrac{4}{3}(z(1-z)/S(z))^2 F(z), \tag{B 14d}$$

$$L_1^{2,0}F(z) = \frac{1}{3S(z)}\Bigg\{\frac{2z(1-z)}{S(z)}[(S'(1)+S'(0))\,z - S'(0)]\,F(z)$$
$$+\frac{d^2}{dz^2}[z^2(1-z)^2F(z)]\Bigg\}, \tag{B 14e}$$

$$L_0^{2,1}F(z) = \tfrac{2}{3}z(1-z)\,\{z(1-z)\,F'(z) + 2(1-2z)\,F(z)\}/S(z). \tag{B 14f}$$

In the expression for $L_1^{1,0}$ we have defined $H(z) = d\{z(1-z)\,F(z)\}/dz$. Equations (B 13) and (B 14) give the operators which are needed for the examples in §6.

In general, it is easy to show by induction from (B 9) and (B 12) that for $k \geqslant 2$

$$L_0^{k,j}F(z) = \begin{cases} \dfrac{((k-1)\,!)^2}{(2k-1)\,!}\,2^{2k-1}\left(\dfrac{z(1-z)}{S(z)}\right)^k F(z) & \text{if} \quad j = 0, \\ \dfrac{((k-2)\,!)^2}{(2k-1)\,(2k-3)\,!}\,2^{2k-3}\left(\dfrac{z(1-z)}{S(z)}\right)^{k-1}\{z(1-z)\,F'(z) & \\ \qquad\qquad + k(1-2z)\,F(z)\} & \text{if} \quad j = 1. \end{cases} \tag{B 15}$$

The result (B 15), when used with the result from (B 13) that $G_0^{k,j} \equiv 0$ for $k \geqslant 2$, shows that, whenever $k \geqslant 2$, I_k^j is $O(\epsilon^{2j-2k})$ as ϵ approaches zero.

REFERENCES

BATCHELOR, G. K. 1970 *J. Fluid Mech.* **44**, 419.
Cox, R. G. 1970 *J. Fluid Mech.* **44**, 791.
Cox, R. G. 1971 *J. Fluid Mech.* **45**, 625.
GEER, J. 1975 *J. Fluid Mech.* **67**, 817.
HANDELSMAN, R. A. & KELLER, J. B. 1967*a* *J. Fluid Mech.* **28**, 131.
HANDELSMAN, R. A. & KELLER, J. B. 1967*b* *SIAM J. Appl. Math.* **15**, 824.
KELLER, J. B. & RUBINOW, S. I. 1976 *J. Fluid Mech.* **75**, 705.
TILLETT, J. P. K. 1970 *J. Fluid Mech.* **44**, 401.